Mathematics 232-B **Spring 2000**

Final Exam

May 10, 2000

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Name

Technology used: ______________________________

Textbook/Notes used: ______________________________

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. Include a careful sketch of any graph obtained by technology in solving a problem. **Only write on one side of each page.**

The Problems

1. Do **two** of the following problems.

   (a) Let V be a vector space and W a **non-empty** subset of V. Suppose that W is closed under both the addition and scalar multiplication of V. Prove that W is a subspace of V.

   (b) An **orthogonal** matrix is an $n \times n$ matrix A satisfying $A^T A = I_n$. Prove the determinant of any orthogonal matrix must be either 1 or -1.

   (c) Do all of the following:

      i. If the rank of a 5×3 matrix A is 3, what is rref(A)?

      ii. If the rank of a 4×4 matrix A is 4, what is rref(A)?

      iii. Consider a linear system of equations $A\overrightarrow{x} = \overrightarrow{b}$, where A is a 4×3 matrix. We are told that $\text{rank}\left[A : \overrightarrow{b}\right] = 4$. How many solutions does this system have?

   (d) Let W be a p-dimensional subspace of $\mathbf{R}^n$. If $\overrightarrow{v}$ is a vector in W for which $\overrightarrow{v}^T\overrightarrow{w} = 0$ for **every** vector $\overrightarrow{w}$ in W, show that $\overrightarrow{v} = \overrightarrow{\theta}$.

2. Do **one** of the following.

   (a) Let $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_m$ be a basis for a subspace V of R^n. Show that $\overrightarrow{x} \in R^n$ is in $V^\perp$ if and only if $\overrightarrow{x}$ orthogonal to the m basis vectors. That is, prove $\overrightarrow{x} \cdot \overrightarrow{v} = \overrightarrow{0}$ for all $\overrightarrow{v} \in V$ if and only if

   $$\overrightarrow{x} \cdot \overrightarrow{v}_i = 0, \quad \text{for } i = 1, \ldots, m.$$

   (b) Let $T : \mathbf{R}^n \longrightarrow \mathbf{R}^m$ be a linear transformation. Suppose that L is the inverse function of T. Show that $L : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ must also be a linear transformation. (You may **not** use the fact that L has a matrix representation until you know that L is linear.)

   (c) Suppose L is the line in R^3 that contains the vector $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and T is the linear transformation $T(\overrightarrow{x}) = A\overrightarrow{x}$ that projects $\overrightarrow{x}$ onto the line L.

i. Use geometric reasoning to explain why the set of vectors, $\mathbf{B} = \left\{ \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} \right\}$ consists of eigenvectors for T with corresponding eigenvalues $1, 0, 0$ respectively. [Hint: the projection of $[-4\ \ 3\ \ 0]^T$ onto L is $\overrightarrow{\theta} = 0\,[-4\ \ 3\ \ 0]^T$ because $[-4\ \ 3\ \ 0]^T$ is perpendicular to $[3\ \ 4\ \ 5]^T$.]

ii. The matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the matrix for T with respect to the ordered eigenbasis in part (a.) in that it satisfies $[T(\overrightarrow{x})]_{\mathbf{B}} = B\,[\overrightarrow{x}]_{\mathbf{B}}$ for every vector $\overrightarrow{x} \in R^3$. What is the matrix S used to give $S^{-1}AS = B$?

iii. Use the above information to find the standard matrix A for the transformation T. That is, find the matrix A for which $T(\overrightarrow{x}) = A\overrightarrow{x}$ for every $\overrightarrow{x}$ in R^3.

3. Do **two** of the following.

(a) Find an orthonormal basis for the null space of

$$A = \begin{bmatrix} 1 & 3 & 10 & 11 & 9 \\ -1 & 2 & 5 & 4 & 1 \\ 2 & -1 & -1 & 1 & 4 \end{bmatrix}.$$

(b) Give a formula for a linear transformation $T : P_3 \longrightarrow P_2$ so that the matrix Q below is the matrix of T with respect to the natural (also known as "standard") bases for P_3 and P_2.

$$Q = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & -1 \end{bmatrix}.$$

(c) Let $S : P_2 \longrightarrow P_3$ be given by $S(p) = x^3 p'' - x^2 p' + 3p$. Find the matrix representation of S with respect to the bases B, C where the basis for P_2 is $B = \{x+1, x+2, x^2\}$ and the basis for P_3 is $C = \{1, x, x^2, x^3\}$.

(d) Let $T : V \longrightarrow V$ be a linear transformation and $B = \{f_1, f_2, f_3, f_4\}$ a basis for V. Find the matrix representation for T with respect to the basis B if $T(f_1) = f_2$, $T(f_2) = f_3$, $T(f_3) = f_1 + f_2$, $T(f_4) = f_1 + 3f_4$.

4. Do **two** of the following.

(a) Find the (real) eigenvalues and eigenspaces of the linear transformation $L : R^{2\times 2} \to R^{2\times 2}$ given by $L(A) = A + A^{T.}$

(b) The set $V = \text{span}\{\cos(t), \sin(t), t\cos(t), t\sin(t)\}$ is an abstract subspace of $C(-\infty, \infty)$. Consider the linear transformation $T : V \to V$ given by

$$T(f) = f'' + f.$$

i. Find a basis for the null space of T.

ii. What is the dimension of the range of T?

(c) Let U, V, W be abstract vector spaces and $S : U \longrightarrow V$, $T : V \longrightarrow W$ linear transformations. Show that the null space of S is contained in the null space of $T \circ S$. That is, $N(S) \subset N(T \circ S)$.

(d) Let $T : V \longrightarrow W$ be a linear transformation. Prove that if T carries linearly independent subsets of V to linearly independent subsets of W, then T must be one-to-one.